Dans cette séance on va voir comment calculer U dans la SVD (2 méthodes) 12.12

③ Calcul de U

Méthode 1 $\longrightarrow$ Complétion + GS

③ (i) $\bar{u}_j := \frac{A\bar{v}_j}{\sigma_j}, \ 1 \leq j \leq l$

(ii) Si $\boxed{l < m}$, on complète $\{\bar{u}_1, \dots, \bar{u}_l\} \subseteq \mathbb{R}^m$ en une base $\{\bar{u}_1, \dots, \bar{u}_l, \bar{w}_{l+1}, \dots, \bar{w}_m\} \subseteq \mathbb{R}^m$ de $\mathbb{R}^m$ et on applique GS pour obtenir une BON $\{\bar{u}_1, \dots, \bar{u}_l, \bar{u}_{l+1}, \dots, \bar{u}_m\} \subseteq \mathbb{R}^m$ de $\mathbb{R}^m$.

$$U := \left[\bar{u}_1 \dots \bar{u}_\ell \; \overline{u_{\ell+1}} \dots \bar{u}_m \right]$$

Exemple 15.9 (fin) Calculer U

$$\bar{u}_1 = \frac{A \cdot \bar{v}_1}{\sigma_1} = \frac{\begin{pmatrix} 3/2\sqrt{2} & 1/2\sqrt{2} \\ 9/10\sqrt{2} & 13/10\sqrt{2} \end{pmatrix} \begin{pmatrix} 4/5 \\ 3/5 \end{pmatrix}}{3/2} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$\bar{u}_2 = \frac{A \cdot \bar{v}_2}{1/2} = 2 \cdot \begin{pmatrix} \frac{3}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ 9/10\sqrt{2} & 13/10\sqrt{2} \end{pmatrix} \begin{pmatrix} -3/5 \\ 4/5 \end{pmatrix} = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$\Rightarrow U = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Fait remarquable : $\sigma_j = \| A \cdot \bar{v}_j \|, \; 1 \leq j \leq \ell$

Méthode 2 $\longrightarrow$ Méthode directe : BON de $\mathrm{Ker}(A^T)$

③ i) $\bar{u}_j := \frac{A\bar{v}_j}{\sigma_j}$, $1 \leq j \leq \ell$

③ ii) Si $\boxed{\ell < m}$, on calcule une base $\mathrm{Ker}(A^T)$ $\{\bar{w}_{\ell+1}, \dots, \bar{w}_m\} \subseteq \mathbb{R}^m$ et on applique GS pour obtenir une BON $\{\bar{u}_{\ell+1}, \dots, \bar{u}_m\}$ de $\mathrm{Ker}(A^T)$.

$$U := \left[\bar{u}_1 \dots \bar{u}_\ell \ \bar{u}_{\ell+1} \dots \bar{u}_m \right]$$

Exemple/Exercice 14.14 Calculer la SVD de

$$A = \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{pmatrix}}_{n=2} \Big] m=3$$

(1) $A^T A = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \longrightarrow \det(A^T A - \lambda I_2) = \lambda(\lambda - 4)$

$$\Longrightarrow \boxed{\underbrace{\lambda_1 = 4}_{} \geq \lambda_2 = 0}$$

$\leftarrow \boxed{\underline{\ell = 1}} = \text{rang}(A)$

$$\Longrightarrow \boxed{\sigma_1 = \sqrt{\lambda_1} = 2}$$

$$\Longrightarrow \Sigma = \underbrace{\begin{pmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}}_{2 \text{ col.}} \Bigg] \text{ 3 lignes}$$

(2) Pour $\boxed{\lambda_1 = 4} \longrightarrow \text{Ker}(A^T A - 4 . I_2) = \text{Ker}\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}$

$$= \text{Ker}\begin{pmatrix} -2 & 2 \\ 0 & 0 \end{pmatrix} = \text{Ker}\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

$$= \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : x_1 = x_2 \right\} = \text{Vect}\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

$$= \text{Vect}\left\{ \underbrace{\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}}_{\bar{v}_1} \right\}$$

Pour $\lambda_2 = 0 \rightsquigarrow \text{Ker}(A^T A) = \text{Vect}\left\{ \underbrace{\begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}}_{\bar{v}_2} \right\}$

$$\Longrightarrow \quad V = \begin{bmatrix} \bar{v}_1 & \bar{v}_2 \end{bmatrix} = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

③ $$\bar{u}_1 = \frac{A\,\bar{v}_1}{\sigma_1} = \frac{\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}}{2} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$$

$$\text{Ker}(A^T) = \text{Ker}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} = \text{Ker}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : x_1 = -x_2 \right\}$$

$$= \left\{ \begin{pmatrix} -x_2 \\ x_2 \\ x_3 \end{pmatrix} : x_2, x_3 \in \mathbb{R} \right\} = \left\{ x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} ; x_2, x_3 \in \mathbb{R} \right\}$$

$$= \mathrm{Vect} \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} = \mathrm{Vect} \underbrace{\left\{ \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}}_{\text{BON.}}$$

$$\Rightarrow \quad U := \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Exemple / Exercice 14.16

Calculer la SVD de

$$A = \underbrace{\begin{pmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{pmatrix}}_{n=3} \Big] m=2$$

Valeurs propres de A^TA:

$$\boxed{\lambda_1 = 12} \geq \boxed{\lambda_2 = 10} \geq \boxed{\lambda_3 = 0}$$

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$$\Sigma = \begin{pmatrix} \sqrt{12} & 0 & 0 \\ 0 & \sqrt{10} & 0 \end{pmatrix} \longleftarrow$$

(2) $$V = \begin{pmatrix} 1/\sqrt{6} & -2/\sqrt{5} & -1/\sqrt{30} \\ 2/\sqrt{6} & 1/\sqrt{5} & -2/\sqrt{30} \\ 1/\sqrt{6} & 0 & 5/\sqrt{30} \end{pmatrix}$$

$$\begin{array}{ccc} \underbrace{\phantom{1/\sqrt{6}}}_{\bar{v}_1} & \underbrace{\phantom{-2/\sqrt{5}}}_{\bar{v}_2} & \underbrace{\phantom{-1/\sqrt{30}}}_{\bar{v}_3} \end{array}$$

(3) $$\bar{u}_1 = \frac{A\bar{v}_1}{\sigma_1} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \qquad \bar{u}_2 = \frac{A\bar{v}_2}{\sigma_2} = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$\Rightarrow \quad U = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$\square$